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LETTER TO THE EDITOR

Jaynes–Cummings model diagonalization based on symmetry

G Benivegna and A Messina

Istituto de Fisica dell'Università degli Studi di Palermo, Gruppo Nazionale del CNR and Centro Interuniversitario di Struttura della Materia del MURST, via Archirafi 36, 90123 Palermo, Italy

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Abstract. We illustrate a new systematic diagonalization procedure through its application to the well known Jaynes–Cummings model. Our method, starting from the consideration of peculiar invariance properties of the Hamiltonian, singles out specific constants of motion by which we succeed in introducing appropriate unitary transformations accomplishing a canonical reduction of the Hamiltonian into a diagonal form. The potentialities of the treatment is briefly discussed.

The interaction between radiation and matter in the well known Jaynes–Cummings (JC) approach [1], is described by a rather simplified but successful model consisting of a single two-level atom coupled, close to resonance, to a single mode of the quantized electromagnetic field. The model is at the heart of laser physics and quantum optics [2]. In addition, the JC model predicts a surprisingly rich dynamics characterized by the occurrence of highly attractive quantum effects [3] some of which may now be observed in the laboratory with the help of experimental techniques involving Rydberg atoms and high-Q microcavities [4].

Exact results concerning the stationary states of the coupled atom-field system were first put forward by Jaynes and Cummings more than thirty years ago, diagonalizing the secular matrix of the total Hamiltonian. These stationary states, known as dressed states of the system [5], identify a new physical unit, named a dressed atom [6], consisting of an atom and the radiation field to which the atom is coupled. The conceptual and formal advantages stemming from the adoption of such a point of view appear particularly evident each time matter-radiation problems such as, for instance, resonance fluorescence [7], are investigated using a canonical transformation approach. In principle, in fact, this way of proceeding should make introducing the dressed atom representation easier, as well as constructing the effective interaction Hamiltonian.

In this letter we present the application to the JC model of a systematic procedure by which we succeed in obtaining, step-by-step, a canonical transformation realizing both the atomic dressing and the separation of pseudospin and bosonic variables. Our method, founded on the existence of symmetry transformations for the basic Hamiltonian of this model, enables a detailed and well-reasoned introduction of appropriate unitary operators accomplishing a canonical reduction of the Hamiltonian to a diagonal form.

The Hamiltonian of the JC model, neglecting the so-called counter-rotating terms, reads ($\hbar = 1$)

$$H_{sf} = \omega\alpha^{\dagger}\alpha + \tfrac{1}{2}\omega_0\sigma_z + \varepsilon(\alpha\sigma_+ + \alpha^{\dagger}\sigma_-) \tag{1}$$

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where the electromagnetic mode is represented by the photon creation and annihilation operators $\alpha^\dagger$ and α, and the atom by the spin-like Pauli operators $\sigma_z, \sigma_+ = \sigma_x + i\sigma_y$ and $\sigma_- = \sigma_x - i\sigma_y$ denoting the inversion, raising and lowering operators respectively. The atom-field coupling strength is measured by ε which is taken to be real and positive. It is easy to verify that for any value of the parameter ξ, the following canonical transformation

$$\begin{cases} \beta = \mathrm{e}^{\mathrm{i}\xi}\alpha \\ \beta^\dagger = \mathrm{e}^{\mathrm{i}\xi}\alpha^\dagger \\ \tilde{\sigma}_+ = \mathrm{e}^{\mathrm{i}\xi}\sigma_+ \\ \tilde{\sigma}_- = \mathrm{e}^{-\mathrm{i}\xi}\sigma_- \\ \tilde{\sigma}_z = \sigma_z \end{cases} \tag{2}$$

is a symmetry transformation for H_{sf}. As it leaves $\alpha^\dagger\alpha$ and σ_z invariant and, in addition, does not mix atomic and field variables, a unitary operator which accomplishes such a transformation may be taken in the form

$$G(\xi) = \exp(-\mathrm{i}\xi N_{sf}) \tag{3}$$

where

$$N_{sf} = \alpha^\dagger\alpha + \tfrac{1}{2}\sigma_z + \tfrac{1}{2} \tag{4}$$

is the operator associated to the total excitation number. The property $[G(\xi), H_{sf}] = 0$ for any $\xi \in R$, implies that N_{sf} is a constant of motion. The first goal of our diagonalization procedure is the decoupling of pseudospin and bosonic degrees of freedom. To this aim we note that the constant of motion obtained from equation (3) for $\xi = \pi$, describing the inversion symmetry of H_{sf} according to (2), has the form

$$G(\pi) = \exp(-\mathrm{i}\pi N_{sf}) = -\sigma_z\cos(\pi\alpha^\dagger\alpha) = G^\dagger(\pi) \tag{5}$$

and is a Hermitian operator with eigenvalues +1 and -1. These properties suggest to seek a unitary operator T satisfying the condition

$$T^\dagger G(\pi) T = -\sigma_z. \tag{6}$$

The reason is that when H_{sf} is transformed by T we obtain a new Hamiltonian operator, commuting with σ_z, and therefore depending on the pseudospin variables only through the same σ_z. This fact becomes evident if one takes into consideration that the Casimir operator relative to the $SU(2)$ group is a multiple of the unit operator. In this way the diagonalization of the transformed Hamiltonian is easily and exactly reduced to that of an effective bosonic Hamiltonian. Writing (6) in the equivalent form $T\sigma_z T^\dagger = \sigma_z\cos(\pi\alpha^\dagger\alpha)$, it is not difficult to guess that T may be chosen as the unitary operator which accomplishes a coordinate rotation about an arbitrary axis on the XY plane, through an angle π or 2π according to the parity of the eigenvalue of $\alpha^\dagger\alpha$. If such an axis is chosen coincident with the X axis, the previous argument leads us to put

$$T = \exp\left(\mathrm{i}\frac{\pi}{2}\alpha^\dagger\alpha\right)\exp\left(-\mathrm{i}\frac{\pi}{2}\alpha^\dagger\alpha\sigma_x\right) \tag{7}$$

where the operator $\exp(\mathrm{i}\frac{\pi}{2}\alpha^\dagger\alpha)$ is inserted at this point only for convenience. Noting that

$$T = \cos^2\left(\frac{\pi}{2}\alpha^\dagger\alpha\right) + \sigma_x \sin^2\left(\frac{\pi}{2}\alpha^\dagger\alpha\right) = T^\dagger \tag{8}$$

we easily deduce

$$T^\dagger G(\pi) T = -\sigma_z \tag{9}$$

$$T^\dagger \alpha T = \alpha\sigma_x \tag{10}$$

$$T^\dagger \sigma_z T = \sigma_z \cos(\pi\alpha^\dagger\alpha) \tag{11}$$

$$T^\dagger \sigma_\pm T = \sigma_\pm \cos^4\left(\frac{\pi}{2}\alpha^\dagger\alpha\right) + \sigma_\mp \sin^4\left(\frac{\pi}{2}\alpha^\dagger\alpha\right) \tag{12}$$

$$T^\dagger(\alpha\sigma_+ \pm \alpha^\dagger\sigma_-)T = (\alpha \pm \alpha^\dagger) - (\alpha \mp \alpha^\dagger)\sigma_z \cos(\pi\alpha^\dagger\alpha). \tag{13}$$

Using (10), (11) and (13) we get the transformed Hamiltonian $H_f \equiv T^\dagger H_{sf} T$ in the following form:

$$H_f = \omega\alpha^\dagger\alpha + \frac{\sigma_z}{2}\cos(\pi\alpha^\dagger\alpha) + \varepsilon(\alpha + \alpha^\dagger) + \varepsilon(\alpha^\dagger - \alpha)\sigma_z\cos(\pi\alpha^\dagger\alpha). \tag{14}$$

The effective bosonic Hamiltonian may be immediately derived from (14) by simply regarding in it σ_z as a c-number coincident with +1 or -1. It should be noted the presence in H_f of the complicated nonlinear last term reflecting the lack of the counter-rotating contributions in the coupling of the old harmonic oscillator and the two-level atom. Of course the unitary operator

$$\tilde{G}(\xi) = T^\dagger G(\xi) T = \exp(-\mathrm{i}\xi N_f) \tag{15}$$

generates, for any ξ, a symmetry transformation for H_f so that

$$N_f = T^\dagger N_{sf} T = \alpha^\dagger\alpha + \tfrac{1}{2}\sigma_z\cos(\pi\alpha^\dagger\alpha) + \tfrac{1}{2} \tag{16}$$

is such that

$$[N_f, H_f] = 0. \tag{17}$$

This last commutation property trivially assures that the following transformation law

$$\gamma = R^\dagger \alpha R \tag{18}$$

$$R = \exp[\mathrm{i}\lambda(H_f - \nu\omega N_f - \tau\omega)] \tag{19}$$

leaves H_f invariant for any choice of the real parameters λ, ν and τ. It is not difficult to convince oneself that the evaluation of γ in terms of α and $\alpha^\dagger$ becomes an easy task once its explicit expression in correspondence to particular values of ν and τ is achieved. We fix the values of ν and τ requiring that R, in each subspace where N_f has a definite value, depends on its exponent only linearly. The possibility of satisfying such a prescription stems from the fact that the invariant subspaces of N_f are, at most, bidimensional, as a direct consequence of the assumed form of the interaction term in the JC Hamiltonian. If we write

$$H_f - \nu\omega N_f - \tau\omega = H'_0 + H_r \tag{20}$$

with

$$H'_0 = \omega\left[(1-\nu)N_f - \frac{1+2\tau}{2}\right] \tag{21}$$

and

$$H_r = \frac{\omega_0-\omega}{2}\sigma_z\cos(\pi\alpha^\dagger\alpha) + \varepsilon(\alpha+\alpha^\dagger) + \varepsilon(\alpha^\dagger-\alpha)\sigma_z\cos(\pi\alpha^\dagger\alpha) \tag{22}$$

taking into account that $[H'_0, H_r] = 0$ and that

$$H_r^2 = \frac{(\omega_0-\omega)^2}{4} + 4\varepsilon^2 N_f. \tag{23}$$

A necessary and sufficient condition to obtain R in the required form is that $\nu = 1$ and $\tau = -\frac{1}{2}$. The correspondent expression of R may be written as

$$R = \mathrm{e}^{\mathrm{i}\lambda H_r} = \cos[\lambda\Phi(\alpha^\dagger\alpha)] + iH_r\Phi^{-1}(\alpha^\dagger\alpha)\sin[\lambda\Phi(\alpha^\dagger\alpha)] \tag{24}$$

where

$$\Phi(\alpha^\dagger\alpha) = \sqrt{\frac{(\omega_0-\omega)^2}{4} + 4\varepsilon^2\left[\alpha^\dagger\alpha + \frac{1}{2}\sigma_z\cos(\pi\alpha^\dagger\alpha) + \frac{1}{2}\right]} \tag{25}$$

provided that the operator $\Phi^{-1}(\alpha^\dagger\alpha)H_r\sin[\lambda\Phi(\alpha^\dagger\alpha)]$ acts as the operator λH_r in the subspace where no excitation is present in the system and at resonance. Using (24) in (18) we finally get the expression of γ for $\nu = 1$ and $\tau = -\frac{1}{2}$:

$$\gamma = 2\alpha + \mathrm{e}^{-\mathrm{i}\lambda H_r}\{\alpha^2 A + [B + 2\cos(\lambda\Phi(\alpha^\dagger\alpha))]\alpha + C\} \tag{26}$$

where

$$A = 2\mathrm{i}\varepsilon\Phi^{-1}(\alpha^\dagger\alpha)\sin[\lambda\Phi(\alpha^\dagger\alpha)] \tag{27}$$

$$B = \cos[\lambda\Phi(\alpha^\dagger\alpha+1)] - \mathrm{i}H_r\Phi^{-1}(\alpha^\dagger\alpha+1)\sin[\lambda\Phi(\alpha^\dagger\alpha+1)] \tag{28}$$

$$C = 2\mathrm{i}\varepsilon N_f\Phi^{-1}(\alpha^\dagger\alpha)\sin[\lambda\Phi(\alpha^\dagger\alpha)]. \tag{29}$$

We have thus established a link between the constant of motion H_r and the existence of that peculiar invariance property of H_f described by (26). Even if we are faced with a quite involved symmetry transformation law, the key intermediate step to carry further the diagonalization of H_{sf} with our treatment is just the knowledge of H_r. In fact, as H_r is Hermitian and the eigenvalues of N_f are all non-negative integers, considering (23), the eigenvalues of H_r must be of the form

$$E_\pm(n) = \pm\sqrt{\frac{(\omega_0-\omega)^2}{4} + 4n\varepsilon^2} \tag{30}$$

with $n = 0, 1, 2, \ldots$. Considering moreover that H_r admits non-diagonal representations in each degenerate subspace of N_f and that its expectation value is negative when no excitation is present in the system, we deduce that $E_\pm(n)$ is always an eigenvalue of H_r with the only exception of $E_+(0)$. If we therefore put

$$\Omega(N_f) = \sqrt{\frac{(\omega_0 - \omega)^2}{4} + 4\varepsilon^2 N_f} \tag{31}$$

the spectrum of the operator $\Omega(N_f)\sigma_z \cos(\pi\alpha^\dagger\alpha)$ coincides with that of H_r. This fact encourages us to look for a unitary operator V, commuting with N_f, and such that

$$V^\dagger H_r V = \Omega(N_f)\sigma_z \cos(\pi\alpha^\dagger\alpha). \tag{32}$$

As a consequence of its definition, V does not commute with $\sigma_z \cos(\pi\alpha^\dagger\alpha)$. This amounts to saying that the operator V cannot be a function of N_f only, and that therefore it must depend on other operators commuting with N_f. Taking into account that $[N_f, T^\dagger(\alpha\sigma_+ \pm \alpha^\dagger\sigma_-)T] = 0$, a look at (13) strongly suggests writing V simply as $\exp[\mathrm{i}\eta(N_f)K_+]$ or $\exp[\eta(N_f)K_-]$, where $\eta(N_f)$ is a Hermitian operator to be determined later and

$$K_\pm = (\alpha \pm \alpha^\dagger) - (\alpha \mp \alpha^\dagger)\sigma_z \cos(\pi\alpha^\dagger\alpha). \tag{33}$$

However, (32) implies $[V, K_+] \neq 0$. This leads us to choose

$$V = \exp\{\eta(N_f)[(\alpha - \alpha^\dagger) - (\alpha + \alpha^\dagger)\sigma_z \cos(\pi\alpha^\dagger\alpha)]\} \tag{34}$$

and to solve the operatorial equation (32) in the unknown $\eta(N_f)$. To this aim we observe that from

$$K_-^2 = -4N_f \tag{35}$$

it is possible to deduce the following representation of V:

$$\begin{aligned} V &= \cosh[\eta(N_f)K_-] + \sinh[\eta(N_f)K_-] \\ &= \cos[\mathrm{i}\eta(N_f)K_-] - \mathrm{i}\sin[\mathrm{i}\eta(N_f)K_-] \\ &= \cos\left[2\sqrt{N_f}\eta(N_f)\right] + \tfrac{1}{2}N_f^{-1/2}K_- \sin\left[2\sqrt{N_f}\eta(N_f)\right] \end{aligned} \tag{36}$$

provided that the operator $N_f^{-1/2}\sin[2\sqrt{N_f}\eta(N_f)]$ acts as $2\eta(N_f)$ in the subspace where no excitation is present in the system. From (36) we immediately get

$$V^\dagger = \cos\vartheta - \tfrac{1}{2}N_f^{-1/2}K_- \sin\vartheta \tag{37}$$

where ϑ is defined as

$$\vartheta = 2\sqrt{N_f}\eta(N_f). \tag{38}$$

We now proceed to the evaluation of $V^\dagger[\sigma_z \cos(\pi\alpha^\dagger\alpha)]V$ and $V^\dagger K_+ V$.

$$\begin{aligned} V^\dagger[\sigma_z \cos(\pi\alpha^\dagger\alpha)]V &= (\cos\vartheta - \tfrac{1}{2}N_f^{-1/2}K_- \sin\vartheta)[\sigma_z \cos(\pi\alpha^\dagger\alpha)](\cos\vartheta + \tfrac{1}{2}N_f^{-1/2}K_- \sin\vartheta) \\ &= [\sigma_z \cos(\pi\alpha^\dagger\alpha)]\cos(2\vartheta) + \tfrac{1}{2}N_f^{-1/2}K_+ \sin(2\vartheta). \end{aligned} \tag{39}$$

Analogously we get:

$$V^\dagger K_+ V = K_+ \cos(2\vartheta) - 2N_f^{1/2} \sin(2\vartheta)[\sigma_z \cos(\pi\alpha^\dagger\alpha)]. \tag{40}$$

Using (39) and (40) in (32) yields the following equation for ϑ:

$$[\tfrac{1}{2}(\omega_0 - \omega)\cos(2\vartheta) - 2\varepsilon N_f^{1/2} \sin(2\vartheta)][\sigma_z \cos(\pi\alpha^\dagger\alpha)] \\ + [\tfrac{1}{4}(\omega_0 - \omega)N_f^{-1/2} \sin(2\vartheta) + \varepsilon\cos(2\vartheta)]K_+ = \Omega(N_f)\sigma_z\cos(\pi\alpha^\dagger\alpha). \tag{41}$$

This condition is equivalent to the following system of equations:

$$\begin{cases} (\omega_0 - \omega)\cos(2\vartheta) - 4\varepsilon N_f^{1/2} \sin(2\vartheta) = 2\Omega(N_f) \\ 4\varepsilon N_f^{1/2} \cos(2\vartheta) + (\omega_0 - \omega)\sin(2\vartheta) = 0 \end{cases} \tag{42}$$

which, at resonance, gives

$$\vartheta = -\pi/4 \tag{43}$$

whereas, out of resonance, can be satisfied by putting

$$\cos(2\vartheta) = \tfrac{1}{2}(\omega_0 - \omega)\Omega^{-1}(N_f) \qquad \sin(2\vartheta) = -2\varepsilon\sqrt{N_f}\Omega^{-1}(N_f). \tag{44}$$

We have thus completed the construction of the unitary operator V realizing the transformation of the constant of motion H_r into the operator $\Omega(N_f)\sigma_z\cos(\pi\alpha^\dagger\alpha)$. Taking into account that $[V, N_f] = 0$, we immediately obtain

$$H = V^\dagger H_f V = \omega(N_f - \tfrac{1}{2}) + \Omega(N_f)\sigma_z\cos(\pi\alpha^\dagger\alpha) \tag{45}$$

which, as anticipated, is in diagonal form. The eigenvalues of H and then of H_{sf} are:

$$E(N, \tilde\sigma) = \omega(N - \tfrac{1}{2}) + \tilde\sigma\sqrt{\tfrac{1}{4}(\omega_0 - \omega)^2 + 4\varepsilon^2 N} \tag{46}$$

where $N = 0, 1, 2, \ldots$ is an eigenvalue of N_f and $\tilde\sigma = \pm 1$ is an eigenvalue of $\sigma_z\cos(\pi\alpha^\dagger\alpha)$. We note that $\tilde\sigma$ is independent on N except when $N = 0$ in which case it assumes only its negative value. The eigenstates of H_{sf} may be written as

$$TV|N, \tilde\sigma\rangle \tag{47}$$

with $|N, \tilde\sigma\rangle$ simultaneous eigenstate of N_f and $\sigma_z\cos(\pi\alpha^\dagger\alpha)$. It is convenient to have explicit expressions of the eigenstates of H_{sf} in terms of the simultaneous eigenstates of $\alpha^\dagger\alpha$ and σ_z which we may denote as $|n, \sigma\rangle$ with $\alpha^\dagger\alpha|n, \sigma\rangle = n|n, \sigma\rangle$ and $\sigma_z|n, \sigma\rangle = \sigma|n, \sigma\rangle$.

We note that

$$|N, \tilde\sigma\rangle = |n = N - \tfrac{1}{2}\tilde\sigma - \tfrac{1}{2} \qquad \sigma = \tilde\sigma\cos(\pi n)\rangle. \tag{48}$$

Using (7), (37) and (48) we succeed in calculating the action of the operator TV on the state $|N, \tilde\sigma\rangle$ obtaining the following result:

$$TV|N, +1\rangle = \cos(\vartheta_N)|n = N - 1, \sigma = +1\rangle + \sin(\vartheta_N)|n = N, \sigma = -1\rangle \tag{49}$$

$$TV|N, -1\rangle = \cos(\vartheta_N)|n = N, \sigma = -1\rangle - \sin(\vartheta_N)n = N - 1, \sigma = +1\rangle \tag{50}$$

for any $N > 0$. When $N = 0$ we get $TV|N = 0, \tilde{\sigma} = -1\rangle = |n = 0, \sigma = -1\rangle$ which is the ground state of the JC Hamiltonian as long as $\varepsilon^2 < \frac{\omega_0\omega}{4}$. The coefficients $\cos(\vartheta_N)$ and $\sin(\vartheta_N)$ are the eigenvalues of the operators $\cos(\vartheta)$ and $\sin(\vartheta)$ deducible from (44). These quantities explicitly expressed in terms of ω, ω_0 and ε are given by

$$\cos(\vartheta_N) = \sqrt{\frac{2\Omega(N) + (\omega_0 - \omega)}{4\Omega(N)}} \tag{51}$$

$$\sin(\vartheta_N) = \sqrt{\frac{2\Omega(N) - (\omega_0 - \omega)}{4\Omega(N)}}. \tag{52}$$

Before closing, we wish to add a few remarks on some characteristic aspects of the treatment presented in this letter. The unifying idea underlying the reduction of H_{sf} to H_f and that of H_f to H is the unitary transformation of a known constant of motion into an operator chosen in such a way to guarantee, *a priori*, an effective simplification in the search of the stationary states of the system. From this point of view the key equations in our treatment are (6) and (32) from which we succeed in getting explicit expressions for T and V. Of course the explicit knowledge of suitable constants of motion, is the main presupposition of such a way of proceeding. A constructive proof of their existence is given in the course of the letter, showing their connection with peculiar symmetry properties of the Hamiltonian.

In conclusion, we believe that the systematic treatment applied in this letter to the JC model, besides the fact that it sheds light on further features of this model, may be useful to investigate other Hamiltonian models such as, for instance, some generalized multimode or multiphoton JC models [8, 9] or the Dicke Hamiltonian [7].

References

[1] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[2] Shore B W and Knight P L 1993 *J. Mod. Opt.* **40** 1195 and references therein
[3] Eberly J H, Narozhny N B and Sanchez-Mondragon J 1980 *Phys. Rev. Lett.* **44** 1323
Narozhny N B, Sanchez-Mondragon J and Eberly J 1981 *Phys. Rev.* A **23** 236
Knight P L and Radmore P M 1982 *Phys. Rev.* A **26** 676
Narozhny N B, Sanchez-Mondragon J and Eberly J H 1983 *Phys. Rev. Lett.* **51** 550
Yoo H I and Eberly J H 1985 *Phys. Rep.* **118** 240
Rempe G, Walther H and Klein N 1987 *Phys. Rev. Lett.* **58** 353
Gea-Banacloche J 1990 *Phys. Rev. Lett.* **65** 3385
Eiselt J and Risken H 1991 *Phys. Rev.* A **43** 346
Dung H D, Tanas R and Shumovsky A S 1991 *J. Mod. Opt.* **38** 2069
Goldberg P and Harrison L C 1991 *Phys. Rev.* A **43** 347
Knight P L and Phoenix S J D 1991 *Phys. Rev. Lett.* **66** 2833
Gea-Banacloche J 1991 *Phys. Rev.* A **44** 5913
Meschede D 1992 *Phys. Rep.* **211** 201
Gea-Banacloche J 1992 *Opt. Commun.* **88** 531
[4] Kleppner D 1981 *Phys. Rev. Lett.* **47** 233
Haroche S 1983 *New Trends in Atomic Physics* ed G Grynberg and R Stora (Amsterdam: North-Holland)
De Martini F *et al* 1991 *Phys. Rev.* A **43** 2480
Bernadot F *et al* 1992 *Europhys. Lett.* **17** 33
Thompson R J, Rempe G and Kimble H J 1992 *Phys. Rev. Lett.* **68** 1132
[5] Cohen Tannoudji C and Reynaud S 1977 *J. Phys. B: At. Mol. Phys.* **10** 345
[6] Cohen Tannoudji C 1968 *Cargese Lectures in Physics* vol 2, ed P Levy, p 347
[7] Leonardi C, Persico F and Vetri G 1986 *Riv. Nuovo Cimento* **9** 1 and references therein
[8] Benivegna G and Messina A 1994 *J. Mod. Opt.* **5** 907

Gou S C 1989 *Phys. Rev.* A **40** 5116
[9] Buck B and Sukumar C V 1981 *Phys. Lett.* **81A** 132
Sukumar C V and Buck B 1981 *Phys. Lett.* **83A** 211
Buck B and Sukumar C V 1984 *J. Phys. A: Math. Gen.* **17** 877
Sukumar C V and Buck B 1984 *J. Phys. A: Math. Gen.* **17** 885
Alsing P and Zubairy M S 1987 *J. Opt. Soc. Am.* B **4** 177
Puri R R and Bullough R K 1988 *J. Opt. Soc. Am.* B **5** 2021
Nasreen T and Razmi M S K 1993 *J. Opt. Soc. Am.* B **10** 1292